

Examples of Special Lagrangian Fibrations.

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§0. Introduction.

Of late there has been a great deal of interest in special Lagrangian submanifolds and manifolds fibred in special Lagrangian submanifolds, motivated by the Strominger-Yau-Zaslow conjecture [33]. One of the basic approaches to finding examples is to exploit symmetries of the ambient manifold. If X is a (non-compact) n -dimensional Calabi-Yau manifold with Kähler form ω and nowhere vanishing holomorphic n -form Ω , and if there is an action of a Lie group G on X preserving these two forms, one can look for G -invariant special Lagrangian submanifolds of X . For us, G will be a torus T^m . Using this sort of symmetry to search for examples reduces the special Lagrangian equations to simpler ones which can be solved.

This technique has been used independently by M. Haskins and D. Joyce ([13] and [21]) to find new examples of special Lagrangian cones and submanifolds, while it has been used independently by E. Goldstein and myself to construct examples of special Lagrangian fibrations on non-compact Calabi-Yau manifolds with T^{n-1} actions. This paper is an extended version of an informally distributed preprint (essentially just the first two sections of this paper) released at the same time as Goldstein's preprint [7]. Goldstein has developed his examples in some very interesting directions somewhat orthogonal to the ones taken here. While there is some overlap between the examples considered here (especially in §2) and those in [7], our goal will be to develop more global information about these fibrations.

The first set of examples, which we discuss in §2, are special Lagrangian fibrations on crepant resolutions of toric Gorenstein singularities. Such examples were already mentioned in [7]. However, earlier, in [10], I gave topological fibrations on such crepant resolutions, with the belief that these would resemble the actual special Lagrangian fibrations on

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these manifolds. What is new here is that we show this is indeed the case; with some mild hypotheses on the Kähler metric ω (which we do not require to be Ricci-flat) we find that the topological construction of §3 of [10] coincides with the special Lagrangian fibrations we give here. This gives us a global understanding of the structure of these fibrations, complementing the results in [7]. We also give a new variant of this construction which yields proper fibrations.

The other main set of examples, which we consider in §3, are smoothings (by flat deformations) of isolated toric Gorenstein singularities. The geometry of such smoothings are controlled by combinatorics of the toric data, by results of K. Altmann [1]. These are new examples of special Lagrangian fibrations not discussed elsewhere in the literature. It is great fun to see how special Lagrangian fibrations change if one starts with a crepant resolution of a toric singularity, degenerates by contracting down to the toric singularity, and then smooths (this process is often called an extremal transition). We discuss some examples of this in §3.

§§4 and 5 are more speculative in nature. In §4, we amplify a brief discussion from [10] about the connections between special Lagrangian fibrations on crepant resolutions of toric singularities and the local mirror symmetry of [4]. In doing so, we make a connection with the work of W.-D. Ruan, who came to the description of torus fibrations on Calabi-Yau hypersurfaces in toric varieties via the dual picture to the fibrations on crepant resolutions. We discuss how one might use an S^1 symmetry to construct special Lagrangian fibrations dual to the fibrations on crepant resolutions. It is likely, however, that this construction will come up against a phenomenon revealed in a very recent preprint of Joyce [22], showing that in the S^1 -invariant case, we might expect to have codimension one rather than codimension two discriminant loci. This is a serious issue for the SYZ conjecture. In the last section, we propose a weaker version of the SYZ conjecture which will hopefully sidestep these issues.

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§1. The Basic Construction.

Recall from [12]:

Definition 1.1. *Let X be a complex n -dimensional manifold, with a Hermitian metric with Kähler form ω and nowhere vanishing holomorphic n -form Ω . Then we say $M \subseteq X$ is special Lagrangian with respect to ω, Ω if $\dim_{\mathbf{R}} M = n$ and $\omega|_M = 0$, $\text{Im } \Omega|_M = 0$.*

Note we do not assume that either $d\omega = 0$ or the metric is Ricci-flat. However, if

the volume form $\omega^n/n!$ is proportional to $\Omega \wedge \bar{\Omega}$, then special Lagrangian submanifolds are volume minimizing as remarked in [8]; see also [6]. Following Joyce, if $d\omega = 0$, we will call the triple (X, ω, Ω) an *almost Calabi-Yau manifold*. Also, we typically want to allow singularities in M ; technically this should be done in the language of currents, but we won't worry about such technicalities. Instead, just view M as being a closed set which is a manifold on an open dense subset of M , and the special Lagrangian condition then is required to hold where M is a manifold.

The main tool we will use for constructing examples of special Lagrangian fibrations will be the following result. This theorem first appeared in print in [7], and similar results appear in [21]. For completeness, we also give the proof here.

Theorem 1.2. *Let (X, ω, Ω) be an almost Calabi-Yau manifold, and suppose there is an effective action of $T := T^m$ on X preserving ω and Ω . Let $\mu_0 : X \rightarrow \mathfrak{t}^* = \mathbf{R}^m$ be the moment map associated to this action.*

- (1) *Let X_1, \dots, X_m be a basis for the vector fields generating the action of T . Then $\Omega_{red} = \iota(X_1, \dots, X_m)\Omega$ descends to an $n - m$ -form on non-singular points of $Z_p := \mu_0^{-1}(p)/T$ for $p \in \mu_0(X)$.*
- (2) *Let ω_{red} be the induced symplectic form on Z_p via symplectic reduction. If $M_{red} \subseteq Z_p$, let M denote the pull-back of M_{red} to $\mu_0^{-1}(p) \subseteq X$. Suppose M is not contained in the set of critical points of μ_0 . Then if $M_{red} \subseteq Z_p$ is special Lagrangian with respect to $\omega_{red}, \Omega_{red}$, $M \subseteq X$ is special Lagrangian with respect to ω, Ω .*
- (3) *Suppose that $g : X \rightarrow Y$ is a continuous map to an $n - m$ -dimensional real manifold Y , satisfying $g(t \cdot x) = g(x)$ for $t \in T$. Then the map $f = (\mu_0, g) : X \rightarrow \mathfrak{t}^* \times Y$ has special Lagrangian fibres with respect to ω, Ω if the induced maps $g : Z_p \rightarrow Y$ have special Lagrangian fibres with respect to $\omega_{red}, \Omega_{red}$ for p in a dense subset of $\mu_0(X)$.*

Proof. Let $x \in \mu_0^{-1}(p)$, and suppose x is not a critical point for μ_0 . Then the tangent space $T_x Z_p$ of Z_p at the point represented by x is identified with

$$\frac{T_x \mu_0^{-1}(p)}{(T_x \mu_0^{-1}(p))^\omega}.$$

Now $(T_x \mu_0^{-1}(p))^\omega$ is the tangent space to the orbit $T \cdot x$, and this is generated by the tangent vectors X_1, \dots, X_m . Thus Ω_{red} vanishes on $(T_x \mu_0^{-1}(p))^\omega$. Since Ω_{red} is invariant under the action of T , Ω_{red} descends to an $n - m$ -form on $T_x Z_p$, proving (1).

Now let $M_{red} \subseteq Z_p$ be special Lagrangian with respect to $\omega_{red}, \Omega_{red}$ with M not contained in the set of critical points of μ_0 . Let $x \in M$ be a regular point for μ_0 .

Let Y_1, \dots, Y_{n-m} be lifts of a basis of tangent vectors to M_{red} at $x \bmod T$. Then $T_x M$ has a basis $X_1, \dots, X_m, Y_1, \dots, Y_{n-m}$. By assumption $\omega(X_i, X_j) = 0$, and $\omega(Y_i, Y_j) = \omega_{red}(Y_i, Y_j) = 0$ since M_{red} is Lagrangian. Finally, since $Y_i \in T_x \mu_0^{-1}(p)$, $\omega(X_i, Y_j) = 0$ since $X_i \in (T_x \mu_0^{-1}(p))^\omega$. Thus $\omega|_{T_x M} = 0$.

To show the fibre is special Lagrangian at x , we just observe that

$$\operatorname{Im} \Omega(X_1, \dots, X_m, Y_1, \dots, Y_{n-m}) = \operatorname{Im} \Omega_{red}(Y_1, \dots, Y_{n-m}) = 0.$$

Thus shows M is special Lagrangian. Item (3) now follows immediately from (2). •

As a basic application of this, let $\varphi(x_1, \dots, x_n)$ be a real-valued function on an open subset of \mathbf{R}^n such that $\varphi(|z_1|^2, \dots, |z_n|^2)$ is pluri-subharmonic on an open subset U of \mathbf{C}^n . Let $\varphi_i = \partial\varphi/\partial x_i$, $\varphi_{ij} = \partial^2\varphi/\partial x_i\partial x_j$. Then the induced Kähler form is

$$\begin{aligned} \omega &= \frac{i}{2} \partial \bar{\partial} \varphi(|z_1|^2, \dots, |z_n|^2) \\ &= \frac{i}{2} \partial \left(\sum_i \varphi_i z_i d\bar{z}_i \right) \\ &= \frac{i}{2} \left(\sum_i \varphi_i dz_i \wedge d\bar{z}_i + \sum_{i,j} \varphi_{ij} \bar{z}_j z_i dz_j \wedge d\bar{z}_i \right). \end{aligned}$$

Corollary 1.3. *The fibres of the map $f : U \rightarrow \mathbf{R}^n$ given by*

$$f = (\varphi_1 |z_1|^2 - \varphi_2 |z_2|^2, \dots, \varphi_1 |z_1|^2 - \varphi_n |z_n|^2, \operatorname{Im}(i^{n+1} \prod_j z_j))$$

are special Lagrangian with respect to ω and $\Omega = dz_1 \wedge \dots \wedge dz_n$.

Proof. ω is invariant under the natural T^n action on \mathbf{C}^n . This T^n action is induced by the vector fields

$$X_j = 2i \left(\bar{z}_j \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial z_j} \right), \quad j = 1, \dots, n.$$

Now

$$\begin{aligned} \iota(X_j)\omega &= \varphi_j(z_j d\bar{z}_j + \bar{z}_j dz_j) \\ &\quad + \sum_i (\varphi_{ij} |z_j|^2 z_i d\bar{z}_i + \varphi_{ij} |z_j|^2 \bar{z}_i dz_i) \\ &= d(\varphi_j |z_j|^2). \end{aligned}$$

Thus X_j is a Hamiltonian vector field with Hamiltonian $\varphi_j|z_j|^2$. In particular, $X_1 - X_2, \dots, X_1 - X_n$ generate a T^{n-1} action with moment map

$$\mu_0 = (f_1, \dots, f_{n-1}) = (\varphi_1|z_1|^2 - \varphi_2|z_2|^2, \dots, \varphi_1|z_1|^2 - \varphi_n|z_n|^2).$$

Furthermore, $f_n = \text{Im}(i^{n+1} \prod z_j)$ is constant on the Hamiltonian trajectories of the first $n-1$ functions. We can now apply Theorem 1.2 with $\mu_0 = (f_1, \dots, f_{n-1})$ and $g = f_n$, as the T^{n-1} -action preserves Ω also. Now

$$\begin{aligned} \iota(X_{f_1}, \dots, X_{f_{n-1}})\Omega &= (-2i)^{n-1} \iota(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \dots, z_1 \frac{\partial}{\partial z_1} - z_n \frac{\partial}{\partial z_n}) dz_1 \wedge \dots \wedge dz_n \\ &= \pm (-2i)^{n-1} d(z_1 \cdots z_n). \end{aligned}$$

It is then clear that g induces a special Lagrangian fibration on the surface $\mu_0^{-1}(p)/T^{n-1}$ for all p . •

§2. Resolutions of Toric Singularities.

Let $N \cong \mathbf{Z}^n$, and let $M = \text{Hom}(N, \mathbf{Z})$ be the dual lattice. Put $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$, $T_{\mathbf{C}}(N) = N \otimes_{\mathbf{Z}} \mathbf{C}^*$, $T(N) = N \otimes_{\mathbf{Z}} \mathbf{R}/N$. Then M can be naturally identified with the group of characters $\text{Hom}(T_{\mathbf{C}}(N), \mathbf{C}^*)$, and we will often identify elements of M with such functions.

Let $\sigma \subseteq N_{\mathbf{R}}$ be a strongly convex rational polyhedral cone. We will assume that σ is a Gorenstein canonical cone. This means that if $n_1, \dots, n_s \in N$ are the set of generators of 1-dimensional faces of σ , then there exists an $m_0 \in M$ such that $\langle m_0, n_i \rangle = 1$ for all i and $\langle m_0, n \rangle \geq 1$ for all $n \in \sigma \cap (N - \{0\})$. Denote by Y_{σ} the corresponding affine toric variety. Y_{σ} has Gorenstein canonical singularities. Let P be the convex hull of n_1, \dots, n_s in the hyperplane $\langle m_0, \cdot \rangle = 1$. From now on we will assume there is a triangulation of P such that the fan Σ obtained as the cone over this triangulation yields a non-singular toric variety Y_{Σ} . Then $Y_{\Sigma} \rightarrow Y_{\sigma}$ is a crepant resolution, and $K_{Y_{\Sigma}} = 0$.

In [10], we constructed a topological fibration on Y_{Σ} . We recall the construction here. Note that $T(N)$ acts naturally on Y_{Σ} . If $N_{m_0} = \{n \in N | \langle m_0, n \rangle = 0\}$, then the subtorus $T(N_{m_0})$ of $T(N)$ also acts on Y_{Σ} . Then one chooses a commutative diagram

$$(2.1) \quad \begin{array}{ccc} Y_{\Sigma} & \xrightarrow{=} & Y_{\Sigma} \\ \downarrow q_1 & & \downarrow \alpha_1 \circ q_1 \\ Y_{\Sigma}/T(N_{m_0}) & \xrightarrow{\alpha_1} & \mathbf{C} \times \mathbf{R}^{n-1} \\ \downarrow q_2 & & \downarrow (z, x) \mapsto (|z|, x) \\ Y_{\Sigma}/T(N) & \xrightarrow{\alpha_2} & \mathbf{R}_{\geq 0} \times \mathbf{R}^{n-1} \end{array}$$

of homeomorphisms α_1, α_2 , with q_1, q_2 the quotient maps. If one composes $\alpha_1 \circ q_1$ with the map $(z, x) \mapsto (\text{Im } z, x)$, one obtains a map $f : Y_\Sigma \rightarrow \mathbf{R}^n$. This is a topological $T^{n-1} \times \mathbf{R}$ fibration, and its discriminant locus was analyzed in [10].

Choose a basis e_1, \dots, e_n of N with dual basis e_1^*, \dots, e_n^* such that $m_0 = e_1^* + \dots + e_n^*$. The dual basis corresponds to coordinates z_1, \dots, z_n on $T_{\mathbf{C}}(N) = N \otimes_{\mathbf{Z}} \mathbf{C}^* \subseteq Y_\Sigma$.

Proposition 2.1. *If Ω is a nowhere vanishing holomorphic n -form on Y_Σ , and $T_{\mathbf{C}}(N)$ is identified with the unique dense orbit of $T_{\mathbf{C}}(N)$ acting on Y_Σ , then $\Omega|_{T_{\mathbf{C}}(N)} = C dz_1 \wedge \dots \wedge dz_n$, where $C \in \mathbf{C}$ is a constant.*

Proof. The proof is standard: see [28]. We give the complete proof here. Clearly $\Omega|_{T_{\mathbf{C}}(N)} = f dz_1 \wedge \dots \wedge dz_n$ for some holomorphic function f . Furthermore, to guarantee that Ω has no zeroes on $T_{\mathbf{C}}(N)$, f must be a monomial, i.e. a constant times a character. It is also easy to check that the expression $dz_1 \wedge \dots \wedge dz_n$ is independent of the choice of basis e_1, \dots, e_n subject to the constraint that $m_0 = e_1^* + \dots + e_n^*$. So we can take e_1, \dots, e_n to be chosen to be edges of an n -dimensional cone τ in Σ ; since Y_Σ is smooth, this forms a basis, and since each e_i then satisfies $\langle m_0, e_i \rangle = 1$, $m_0 = e_1^* + \dots + e_n^*$. Then τ^\vee is generated by e_1^*, \dots, e_n^* , so Y_Σ contains an open affine subset $\text{Spec } \mathbf{C}[\tau^\vee \cap M] = \text{Spec } \mathbf{C}[z_1, \dots, z_n]$. Then $f dz_1 \wedge \dots \wedge dz_n$ extends to a non-zero n -form on this open affine subset if and only if f is constant, as desired. •

Theorem 2.2. *Let ω be the Kähler form of a Kähler metric on Y_Σ , invariant under the action of $T(N)$. Let Ω be the nowhere vanishing n -form on Y_Σ which restricts to $dz_1 \wedge \dots \wedge dz_n$ on $T_{\mathbf{C}}(N)$. Let $\mu : Y_\Sigma \rightarrow \mathbf{R}^n$ be the moment map associated to this $T(N)$ action, and let $\mu_0 : Y_\Sigma \rightarrow \mathbf{R}^{n-1}$ be the moment map associated to the $T(N_{m_0})$ action. Then the function $g : T_{\mathbf{C}}(N) \rightarrow \mathbf{R}$ given by*

$$g(z_1, \dots, z_n) = \text{Im } i^{n+1} \prod z_i$$

extends to a map $g : Y_\Sigma \rightarrow \mathbf{R}$. Furthermore, $f = (g, \mu_0) : Y_\Sigma \rightarrow \mathbf{R}^n$ is a special Lagrangian fibration with respect to ω, Ω . If μ is proper, then f coincides topologically with the construction given in [10], §3.

Proof. We can identify m_0 with the character $\prod z_i$ on $T_{\mathbf{C}}(N)$. A priori m_0 extends to only a rational function on Y_Σ . In fact, it extends to a regular function. To show this, we need to show it extends across every prime divisor of Y_Σ contained in $Y_\Sigma - T_{\mathbf{C}}(N)$. Let n generate a ray in the fan Σ , corresponding to some such divisor D_n . The dual cone to $\tau = \mathbf{R}_{\geq 0}n$ is the half-plane

$$\tau^\vee = \{m \in M \mid \langle m, n \rangle \geq 0\}.$$

Now $\langle m_0, n \rangle = 1$, so $m_0 \in \tau^\vee$, $-m_0 \notin \tau^\vee$. The open affine piece of Y_Σ corresponding to the cone τ is $\text{Spec } \mathbf{C}[\tau^\vee \cap M]$. Thus m_0 is a regular function on this open set. Since this open set contains a dense subset of D_n , m_0 extends across D_n , and in fact takes the value zero on D_n , since $-m_0$ (corresponding to the character $\prod z_j^{-1}$) is not in τ^\vee .

Thus $i^{n+1}m_0$ gives a map $Y_\Sigma \rightarrow \mathbf{C}$, from which the first claim follows. Using the moment map μ_0 of the $T(N_{m_0})$ -action, we obtain a map $(i^{n+1}m_0, \mu_0) : Y_\Sigma \rightarrow \mathbf{C} \times \mathbf{R}^{n-1}$. Composing this map with $(z, x) \mapsto (\text{Im } z, x)$, we obtain $f : Y_\Sigma \rightarrow \mathbf{R}^n$, an extension of the special Lagrangian fibration $f : T_{\mathbf{C}}(N) \rightarrow \mathbf{R}^n$ arising in Corollary 1.3.

To finish, we show f coincides with the construction of [10] if μ is proper.

Because we are assuming μ is proper, the following facts follow from [15] Theorem 4.1: $\mu(Y_\Sigma)$ is convex, and μ has connected fibres. Thus, in particular, μ identifies $\mu(Y_\Sigma)$ with $Y_\Sigma/T(N)$. Furthermore, $\mu(Y_\Sigma)$ is a closed, locally polyhedral convex set, and the extremal points of $\mu(Y_\Sigma)$ are images of fixed points of the $T(N)$ -action. Finally the tangent “wedge” to $\mu(Y_\Sigma)$ at such a point $\mu(x)$ is generated by the weights of the $T(N)$ -action on $T_x Y_\Sigma$. There is a 1-1 correspondence between fixed points of the $T(N)$ -action on Y_Σ and maximal cones τ of the fan Σ . Now since Y_Σ is non-singular, each such cone τ is generated by a basis v_1, \dots, v_n of N , and the weights of the $T(N)$ -representation on $T_x Y_\Sigma$, x the point corresponding to τ , are v_1^*, \dots, v_n^* . Furthermore, since $\langle m_0, v_i \rangle = 1$, it follows that $m_0 = \sum v_i^*$. In particular, m_0 is in the interior of the tangent “wedge” of each such extremal point.

Let $r : \mathbf{R}^n \rightarrow \mathbf{R}^{n-1}$ be given by $(x_1, \dots, x_n) \mapsto (x_1 - x_2, \dots, x_1 - x_n)$. The composition $r \circ \mu : Y_\Sigma \rightarrow \mathbf{R}^{n-1}$ is the moment map μ_0 of the $T(N_{m_0})$ action on Y_Σ .

We will now show that if $L = r^{-1}(c)$, $c \in \mathbf{R}^{n-1}$, is a line in $\mathbf{R}^n = M \otimes_{\mathbf{Z}} \mathbf{R}$ parallel to m_0 , then $L \cap \mu(Y_\Sigma)$ is a ray. Indeed, if $L \cap \partial\mu(Y_\Sigma)$ is non-empty, the description above of the tangent wedges of the extremal points shows that $L \cap \partial\mu(Y_\Sigma)$ consists of one point. Thus $L \cap \mu(Y_\Sigma)$ is closed (as μ is proper) and has one boundary point, so it is a ray. If $L \cap \partial\mu(Y_\Sigma)$ is empty, then either $L \cap \mu(Y_\Sigma) = \emptyset$ or $L \subseteq \mu(Y_\Sigma)$. In either case, choose a line $l \subseteq \mathbf{R}^{n-1}$ such that the plane $r^{-1}(l)$ contains L , $r^{-1}(l) \cap \mu(Y_\Sigma) \neq \emptyset$, and $r^{-1}(l) \not\subseteq \mu(Y_\Sigma)$. Then $S = r^{-1}(l) \cap \mu(Y_\Sigma)$ is a closed convex set. It can only contain L if S is a half-plane with edge parallel to L , contradicting the description of the tangent wedges to $\mu(Y_\Sigma)$. If $L \cap S = \emptyset$, then there is a supporting line to S , L' , parallel to L . This also contradicts the description of the tangent wedges. Thus $L \cap \mu(Y_\Sigma)$ is a ray.

We now replace r with its restriction to $\mu(Y_\Sigma)$. We have $r : \mu(Y_\Sigma) \rightarrow \mathbf{R}^{n-1}$ is surjective, with each fibre being a ray parallel to m_0 .

We now wish to define homeomorphisms α_1, α_2 as in the diagram (2.1) so that $f = \text{Im} \circ \alpha_1 \circ q_1$, where $\text{Im} : \mathbf{C} \times \mathbf{R}^{n-1} \rightarrow \mathbf{R}^n$ is the map $(z, x) \mapsto (\text{Im } z, x)$. This will show that f coincides with the construction of [10], §3.

Note that the value of $|m_0|$ only depends on the $T(N)$ -orbit of (z_1, \dots, z_n) , so $|m_0|$ descends to a map $|m_0| : \mu(Y_\Sigma) \rightarrow \mathbf{R}_{\geq 0}$. We define α_2 as the product map $(|m_0|, r) : \mu(Y_\Sigma) \rightarrow \mathbf{R}_{\geq 0} \times \mathbf{R}^{n-1}$. To show that α_2 is a homeomorphism, it is enough to show that $|m_0| : r^{-1}(c) \rightarrow \mathbf{R}_{\geq 0}$ for $c \in \mathbf{R}^{n-1}$ is a homeomorphism. Now $r^{-1}(c)$ is a ray whose endpoint is in $\partial\mu(Y_\Sigma)$, and $\mu^{-1}(\partial\mu(Y_\Sigma)) = Y_\Sigma - T_{\mathbf{C}}(N)$. Thus $|m_0|$ takes the value zero on the endpoint of $r^{-1}(c)$. So all we need to know is that $|m_0|$ is monotonically increasing on $r^{-1}(c)$, and goes to ∞ .

To see this, note that $Z_c := \mu_0^{-1}(c)/T(N_0)$ is in fact isomorphic to \mathbf{C} , and m_0 descends to a holomorphic function on Z_c . The maximum modulus theorem then tells us that $|m_0|$ is monotonically increasing on $r^{-1}(c)$, and Liouville's theorem tells us $|m_0|$ goes to ∞ .

Thus α_2 is an homeomorphism.

Similarly, since $(i^{n+1}m_0, \mu_0)$ is constant on $T(N_{m_0})$ -orbits, it descends to give a map $\alpha_1 : Y_\Sigma/T(N_{m_0}) \rightarrow \mathbf{C} \times \mathbf{R}^{n-1}$. It is easy to check now that α_1 is a homeomorphism making the diagram (2.1) commute, and that $f = \text{Im} \circ \alpha_1 \circ q_1$, as desired. •

We next comment as to when the hypotheses of Theorem 2.2 can be achieved.

First, we recall the standard construction of Y_Σ as a symplectic quotient: see [3] for details. Here as always Y_Σ is assumed to be non-singular. Let $\Sigma(1)$ denote the set of one-dimensional faces of σ . For each $\sigma \in \Sigma(1)$ there is a toric divisor D_σ corresponding to σ , and $\{D_\sigma | \sigma \in \Sigma(1)\}$ generates $\text{Pic}Y_\Sigma$. Define a map

$$\mathbf{Z}^{\Sigma(1)} \xrightarrow{\pi} N$$

taking a standard basis vector e_σ of $\mathbf{Z}^{\Sigma(1)}$, $\sigma \in \Sigma(1)$, to the generator of the corresponding one-dimensional face of Σ . Let $K = \ker \pi$, and assume (as will always be the case in our examples of interest) that π is surjective. For $I \subseteq \Sigma(1)$, define $e_I \subseteq \mathbf{C}^{\Sigma(1)}$ by

$$\{(z_j)_{j \in \Sigma(1)} | z_i = 0 \text{ for } i \notin I\}$$

and let $\bar{I} = \Sigma(1) \setminus I$. Let

$$S = \{I \subseteq \Sigma(1) | \bar{I} \text{ does not span a cone in } \Sigma\}.$$

Define $\mathcal{U}_\Sigma \subseteq \mathbf{C}^{\Sigma(1)}$ by

$$\mathcal{U}_\Sigma = \mathbf{C}^{\Sigma(1)} \setminus \bigcup_{I \in S} e_I.$$

\mathcal{U}_Σ inherits the standard Kähler form

$$\omega = \frac{i}{2} \sum_{j \in \Sigma(1)} dz_j \wedge d\bar{z}_j.$$

Let \mathfrak{t} denote the Lie algebra of $T(\mathbf{Z}^{\Sigma(1)})$ and \mathfrak{k} denote the Lie algebra of $T(K)$. There is a well-known correspondence between elements of \mathfrak{t}^* and real divisors: an element $\alpha \in \mathfrak{t}^*$ corresponds to the real divisor

$$D_\alpha = \sum_{\sigma \in \Sigma(1)} \alpha(e_\sigma) D_\sigma.$$

On the other hand, such an α defines a piecewise linear function on the support of the fan Σ . Indeed, for each cone τ of Σ , choose $m_\tau \in M$ so that $\langle m_\tau, \pi(e_\sigma) \rangle = -\alpha(e_\sigma)$ for each one-dimensional face σ of τ . Then define $\phi_\alpha : |\Sigma| \rightarrow \mathbf{R}$ by $\phi_\alpha(x) = \langle m_\tau, x \rangle$ for $x \in \tau$. Then the \mathbf{R} -divisor D_α is ample if and only if ϕ_α is strictly upper convex, i.e.

$$\langle m_\tau, \pi(e_\sigma) \rangle > -\alpha(e_\sigma)$$

whenever σ is not a face of an n -dimensional cone τ . Two divisors are linearly equivalent if the corresponding piecewise linear functions differ by a linear function; thus $\text{Pic} Y_\Sigma \otimes \mathbf{R} \cong \mathfrak{k}^*$ naturally via the projection $p : \mathfrak{t}^* \rightarrow \mathfrak{k}^*$.

Now the action of $T(\mathbf{Z}^{\Sigma(1)})$ on \mathcal{U}_Σ induces the standard moment map $\nu : \mathcal{U}_\Sigma \rightarrow \mathfrak{t}^*$ and $\nu' = p \circ \nu : \mathcal{U}_\Sigma \rightarrow \mathfrak{k}^*$ is the moment map for the $T(K)$ action on \mathcal{U}_Σ .

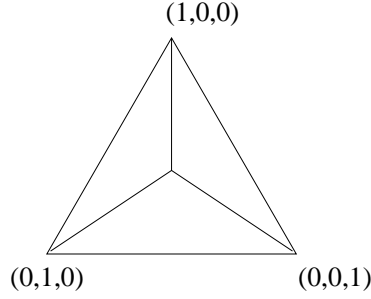
Let $\alpha \in \mathfrak{k}^*$ be the class of an ample \mathbf{R} -divisor on Y_Σ . Then by Proposition 3.1.1 of [3], $\nu'^{-1}(\alpha)/T(K)$ is homeomorphic to Y_Σ , and the induced, reduced symplectic form ω_α on Y_Σ has cohomology class equal to D_α . This symplectic form is a Kähler form.

The residual $T(N)$ action on Y_Σ preserves this Kähler form, so this gives an example of a $T(N)$ -invariant Kähler form in each Kähler class. Furthermore, this action induces the moment map $\mu : Y_\Sigma \rightarrow p^{-1}(\alpha)$, the latter being a translation of \mathfrak{n}^* , where \mathfrak{n} is the Lie algebra of $T(N)$. Here we use the exact sequence

$$0 \longrightarrow \mathfrak{n}^* \longrightarrow \mathfrak{t}^* \xrightarrow{p} \mathfrak{k}^* \longrightarrow 0.$$

By [3], §3.2, the image of μ is $P_\alpha = p^{-1}(\alpha) \cap \mu(\mathcal{U}_\Sigma)$. It is then not difficult to see that P_α is closed and $\mu : Y_\Sigma \rightarrow \mathfrak{n}^*$ is proper. Furthermore, knowing the image of μ allows us to determine the discriminant locus of the special Lagrangian fibration precisely.

Example 2.3. Let $N = \mathbf{Z}^3 + \frac{1}{3}(1, 1, 1)$, and let σ be the cone spanned by $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Σ is then obtained by subdividing σ at $\frac{1}{3}(1, 1, 1)$, giving



Then

$$\Sigma(1) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), \frac{1}{3}(1, 1, 1)\}$$

and the matrix for π is

$$\begin{pmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{pmatrix}$$

with kernel generated by $(1, 1, 1, -3)$. Thus

$$\mathcal{U}_\Sigma = \mathbf{C}^4 \setminus \{z_1 = z_2 = z_3 = 0\}.$$

The moment map ν' is given by

$$\nu'(z_1, z_2, z_3, z_4) = |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2.$$

For $a > 0$, let $(Y_\Sigma, \omega_\Sigma)$ be given by $\nu'^{-1}(a)/T(K)$. Now

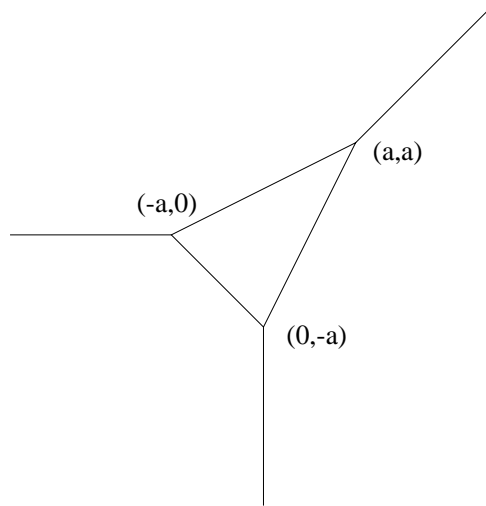
$$p^{-1}(a) = \{(r_1, r_2, r_3, (r_1 + r_2 + r_3 - a)/3) | r_1, r_2, r_3 \in \mathbf{R}\},$$

so P_a can be identified with the set

$$\{(r_1, r_2, r_3) \in \mathbf{R}^3 | r_1, r_2, r_3 \geq 0, \quad r_1 + r_2 + r_3 \geq a\}.$$

Finally, the discriminant locus of the induced special Lagrangian fibration $f = (g, \mu_0) : Y_\Sigma \rightarrow \mathbf{R}^3$ is the planar graph which is the image of the 1-skeleton of the boundary of P_a

under the projection $\mathbf{R}^3 \rightarrow \mathbf{R}^2$ given by $(r_1, r_2, r_3) \mapsto (r_1 - r_2, r_1 - r_3)$. This is



In these examples, the moment map μ is always proper. More generally, however, this need not be the case. Since the above metrics are not, in general, Ricci-flat, we might be interested in a wider range of metrics. Now in any event the action of $T(N)$ is induced by the action of $T_{\mathbf{C}}(N)$, so it follows from [14], Convexity Theorem, §7, that the image of the moment map μ is convex with connected fibres, so μ identifies $\mu(Y_{\Sigma})$ with $Y_{\Sigma}/T(N)$ in general. However, $\mu(Y_{\Sigma})$ may not be closed, in which case μ is not proper. Thus we need some additional asymptotic conditions on ω .

For example in [34,35], existence of complete Ricci-flat metrics on some non-compact manifolds was proven. More precise information in certain cases was given in [19,20], where it was proved that there exists ALE or quasiALE Ricci-flat metrics on crepant resolutions of \mathbf{C}^n/G , with $G \subseteq SU(n)$. If G is abelian, then \mathbf{C}^n/G is toric. By the uniqueness results of [19,20], these metrics are invariant under the induced $T(N)$ action, and the above theorem applies. Furthermore, the ALE or quasi-ALE conditions guarantee the moment map is proper, being asymptotic to the moment map with respect to the Euclidean metric.

In general, it is not known when Ricci-flat metrics exist on Y_{Σ} for general Gorenstein cones σ . However, in some cases it is possible to use the results of [34,35] to find further examples. In any event, as long as the moment map with respect to ω is proper, its image will coincide with the image of the moment map induced by ω' given by the symplectic reduction method above when the classes $[\omega], [\omega'] \in H^2(Y_{\Sigma}, \mathbf{R})$ coincide. Thus we always get a precise description of the discriminant locus.

Since we are interested in special Lagrangian fibrations because of the SYZ conjecture, it is actually more interesting to construct proper special Lagrangian fibrations. In [10],

Remark 3.5, we noted we can construct a topological “properification” of the map $f : Y_\Sigma \rightarrow \mathbf{R}^3$. Here, we take an alternative route and construct proper special Lagrangian fibrations as a generalization of [8], Example 1.2.

Theorem 2.4. *Let ω be the Kähler form of a Kähler metric on Y_Σ , invariant under the action of $T(N)$. Let Ω be the holomorphic n -form on Y_Σ which restricts to $dz_1 \wedge \cdots \wedge dz_n$ on $T_{\mathbf{C}}(N)$. Let $\mu : Y_\Sigma \rightarrow \mathbf{R}^n$ be the moment map of the $T(N)$ action, and assume μ is proper. Let $\mu_0 : Y_\Sigma \rightarrow \mathbf{R}^{n-1}$ be the moment map associated to the $T(N_{m_0})$ -action. Set*

$$Y'_\Sigma = Y_\Sigma \setminus \{1 + \prod_{i=1}^n z_i = 0\}.$$

Here $\prod_{i=1}^n z_i$ is identified with the character m_0 , and as such, defines a regular function on Y_Σ , as we saw in the proof of Theorem 2.2. Let

$$\Omega' = \frac{\Omega}{i^n(1 + \prod_{i=1}^n z_i)}$$

be a nowhere vanishing holomorphic n -form on Y'_Σ . Then $f' = (\log |1 + \prod_{i=1}^n z_i|, \mu_0) : Y'_\Sigma \rightarrow \mathbf{R}^n$ is a proper special Lagrangian fibration with respect to ω, Ω' , with the same discriminant locus as that of $f : Y_\Sigma \rightarrow \mathbf{R}^n$ constructed in Theorem 2.2. Furthermore, the general fibre is an n -torus.

Proof. Let $\mu_0 = (f_1, \dots, f_{n-1})$ be as in the proof of Corollary 1.3. Then

$$\begin{aligned} \iota(X_{f_1}, \dots, X_{f_{n-1}})\Omega' &= \pm \frac{(-2i)^{n-1} d(z_1 \cdots \cdots z_n)}{i^n(1 + \prod_{i=1}^n z_i)} \\ &= \pm (-2)^{n-1} d(i \log(1 + \prod_{i=1}^n z_i)), \end{aligned}$$

so by Theorem 1.2,

$$f' = (\text{Im}(i \log(1 + \prod_{i=1}^n z_i)), \mu_0) = (\log |1 + \prod_{i=1}^n z_i|, \mu_0)$$

is a special Lagrangian fibration with respect to ω, Ω' .

Now consider $c \in \mathbf{R}^{n-1}$ and $\mu_0^{-1}(c)/T(N_{m_0})$, with $\mu_0 : Y_\Sigma \rightarrow \mathbf{R}^{n-1}$ (rather than its restriction to Y'_Σ). Now in the proof of Theorem 2.2, it was shown that m_0 (as a regular function) descended to $\mu_0^{-1}(c)/T(N_{m_0})$ to give an isomorphism $m_0 : \mu_0^{-1}(c)/T(N_{m_0}) \rightarrow \mathbf{C}$. Then m_0 induces an isomorphism $m_0 : (\mu_0^{-1}(c) \cap Y'_\Sigma)/T(N_{m_0}) \rightarrow \mathbf{C} \setminus \{-1\}$. Furthermore

$\log |1 + m_0|$ gives an S^1 -fibration $(\mu_0^{-1}(c) \cap Y'_\Sigma)/T(N_{m_0}) \rightarrow \mathbf{R}$. The inverse image of the general fibre of this map in $\mu_0^{-1}(c)$ is then $T^{n-1} \times S^1 = T^n$. Thus the general fibre of f' is an n -torus. Now orbits of $T(N_{m_0})$ only drop dimension when $m_0 = 0$. But when $m_0 = 0$, $\log |1 + m_0| = 0$, and this makes it clear the discriminant locus coincides with that of f . •

Remark 2.5. (1) While Y'_Σ is certainly not a holomorphic partial compactification of Y_Σ (being contained in Y_Σ), it is possible to prove $Y'_\Sigma \rightarrow \mathbf{R}^n$ does coincide with a topological partial compactification of $Y_\Sigma \rightarrow \mathbf{R}^n$.

(2) All singular fibres $f'^{-1}(b)$ of the above fibration have the same basic structure: there is a fibration $f'^{-1}(b) \rightarrow S^1$ with all but one fibre a T^{n-1} , with the remaining fibre being a torus of dimension between 0 and $n - 2$.

(3) Unlike the non-proper case, we should not expect there to exist a complete Kähler metric ω on Y'_Σ satisfying ω^n proportional to $\Omega' \wedge \bar{\Omega}'$.

§3. Deformations of toric singularities.

The deformation theory of toric Gorenstein singularities is controlled by the combinatorics of the corresponding cones. In particular, if Y_σ is an isolated toric singularity, then [1] gives a beautiful description of the versal deformation space of Y_σ . Irreducible components of this versal deformation space are in one-to-one correspondence with maximal Minkowski decompositions of the polytope P (P as in §2, with σ the cone over P).

Altmann's construction is as follows. Let N , M and σ be as in §2, and assume Y_σ has only an isolated singularity. Now σ is a cone over a polytope P contained in the affine hyperplane $\langle m_0, \cdot \rangle = 1$. By choosing some element n_0 such that $\langle m_0, n_0 \rangle = 1$, we can identify P with $P - n_0$ in the hyperplane $L_{\mathbf{R}} \subseteq N_{\mathbf{R}}$ given by $L = m_0^\perp$. Let $P = R_0 + \cdots + R_p$ be a Minkowski decomposition of P inside $L_{\mathbf{R}}$. What this means is that R_0, \dots, R_p are convex subsets of L such that

$$P = \{r_0 + \cdots + r_p \mid r_i \in R_i\}.$$

Example 3.1. We focus on the prettiest example, a cone over a del Pezzo surface of degree 6. We can take σ to be generated by

$$n_1, \dots, n_6 = (0, 0, 1), (1, 0, 1), (2, 1, 1), (2, 2, 1), (1, 2, 1), (0, 1, 1)$$

so that P in \mathbf{R}^2 is

$$\begin{array}{c}
 \text{Diagram of } P \text{ (a parallelogram)} = \begin{array}{c} (0,1) \\ | \\ (0,0) \end{array} + \begin{array}{c} \text{---} \\ (0,0) \quad (1,0) \end{array} + \begin{array}{c} (1,1) \\ / \\ (0,0) \end{array} \\
 \\
 \text{or} \quad \begin{array}{c} (1,1) \\ \triangle \\ (0,0) \quad (1,0) \end{array} + \begin{array}{c} (0,1) \quad (1,1) \\ \triangle \\ (0,0) \end{array}
 \end{array}$$

with two different Minkowski decompositions.

Now to each such Minkowski decomposition $P = R_0 + \cdots + R_p$, Altmann constructs a flat deformation of Y_σ as follows. Let $N' = L \oplus \mathbf{Z}^{p+1}$, and let e_0, \dots, e_p denote the standard basis of \mathbf{Z}^{p+1} and define

$$\tilde{\sigma} = \text{Cone} \left(\bigcup_{k=0}^p (R_k \times \{e_k\}) \right) \subseteq N'_{\mathbf{R}}$$

where $\text{Cone}(S)$ denotes the cone generated by the set $S \subseteq N'_{\mathbf{R}}$. If one writes $N = L \oplus \mathbf{Z}n_0$, there is a diagonal embedding $N \hookrightarrow N'$ given by $l + an_0 \mapsto l + a(e_0 + \cdots + e_p)$. Under this embedding $\sigma = \tilde{\sigma} \cap N_{\mathbf{R}}$, and hence we obtain a closed embedding $Y_\sigma \hookrightarrow Y_{\tilde{\sigma}}$. On the other hand, under the projection $N' \rightarrow \mathbf{Z}^{p+1}$, $\tilde{\sigma}$ maps to the cone generated by e_0, \dots, e_p , and this induces a morphism $Y_{\tilde{\sigma}} \rightarrow \mathbf{C}^{p+1}$. Altmann proves the composed morphism $f : Y_{\tilde{\sigma}} \rightarrow \mathbf{C}^{p+1}/\mathbf{C}(1, \dots, 1)$ is a flat deformation of Y_σ , with $f^{-1}(0) = Y_\sigma$. More explicitly, the surjection $N' \rightarrow \mathbf{Z}^{p+1}$ gives an inclusion $\mathbf{Z}^{p+1} \hookrightarrow M'$, with e_0^*, \dots, e_p^* mapping to elements of M' corresponding to characters t_0, \dots, t_p . These characters extend to regular functions on $Y_{\tilde{\sigma}}$, and f is given by $(t_0 - t_1, \dots, t_0 - t_p)$.

The main point for us then is that the functions t_0, \dots, t_p are invariant under the action of $T_{\mathbf{C}}(L) \subseteq T_{\mathbf{C}}(N')$, and thus $T_{\mathbf{C}}(L)$ acts on the fibres of f . This gives the desired T^{n-1} -action on deformations of Y_σ .

What about a $T(L)$ -invariant holomorphic n -form on the fibres of f ? Well note that $\tilde{\sigma}$ is a Gorenstein canonical cone; if $m'_0 = e_0^* + \cdots + e_p^* \in M'$, then all generators of $\tilde{\sigma}$ evaluate to 1 on m'_0 . Thus there is a nowhere vanishing holomorphic $n + p$ -form Ω on the smooth part of $Y_{\tilde{\sigma}}$ (whose restriction to $T_{\mathbf{C}}(N') \subseteq Y_{\tilde{\sigma}}$ is described by Proposition 2.1). We then have

Proposition 3.2. *If t_0, \dots, t_p are coordinates on \mathbf{C}^{p+1} , and $\partial_{t_0}, \dots, \partial_{t_p}$ are lifts of the corresponding vector fields to $Y_{\tilde{\sigma}}$, then for $x \in \mathbf{C}^{p+1}/\mathbf{C}(1, \dots, 1)$,*

$$\Omega_x = (\iota(\partial_{t_1}, \dots, \partial_{t_p})\Omega)|_{f^{-1}(x)}$$

is a well-defined nowhere vanishing holomorphic n -form on the non-singular part of $f^{-1}(x)$, which we write as $Y_{\tilde{\sigma},x}^{ns}$. In addition, Ω_x is $T(L)$ -invariant. Finally, if z_1, \dots, z_{n-1} are a basis of characters for $T_{\mathbf{C}}(L)$, then $z_1, \dots, z_{n-1}, t_0, \dots, t_p$ form a basis of characters for $T_{\mathbf{C}}(N')$ and

$$\Omega|_{T_{\mathbf{C}}(N')} = \frac{dz_1 \wedge \dots \wedge dz_{n-1}}{\prod z_i} \wedge dt_0 \wedge \dots \wedge dt_p$$

so up to sign

$$\Omega_x|_{T_{\mathbf{C}}(N) \cap f^{-1}(x)} = \frac{dz_1 \wedge \dots \wedge dz_{n-1}}{\prod z_i} \wedge dt_0.$$

Proof. That Ω_x is well-defined, independent of the lifts of the ∂_{t_i} 's is standard, and since $\partial_{t_1}, \dots, \partial_{t_p}$ are linearly independent at a non-singular point of $f^{-1}(x)$, Ω_x is non-zero. Also, Ω is invariant under the action of $T(m'_0{}^\perp)$, and $L \subseteq m'_0{}^\perp$, so Ω is invariant under $T(L)$. Since t_1, \dots, t_p are also invariant under $T(L)$, so is Ω_x . Finally, the explicit form for Ω follows from Proposition 2.1 and the explicit value for m'_0 . •

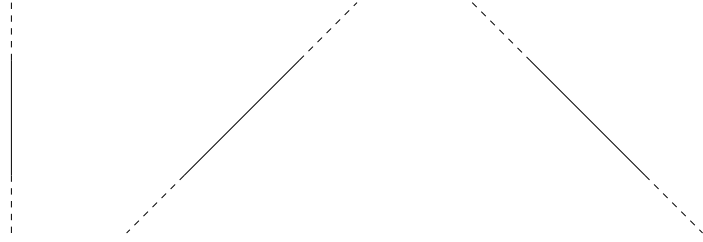
Proposition 3.3. *If ω is a $T(L)$ -invariant Kähler form on $Y_{\tilde{\sigma},x}^{ns}$, let $\mu : Y_{\tilde{\sigma},x}^{ns} \rightarrow L_{\mathbf{R}}^*$ be the moment map associated to the $T(L)$ -action. Then $f : Y_{\tilde{\sigma},x}^{ns} \rightarrow \mathbf{R} \times L_{\mathbf{R}}^*$ given by $f = (\text{Im}(i^{n+1}t_0), \mu)$ is a special Lagrangian fibration. Furthermore, f is surjective if μ is and the general fibre is diffeomorphic to $\mathbf{R} \times T(L)$. If x is represented by $(x_0, \dots, x_p) \in \mathbf{C}^{p+1}$, then the discriminant locus is contained in the union of $p+1$ hyperplanes $\{\text{Im}(i^{n+1}(x_0 - x_k)) | k = 0, \dots, p\} \times L_{\mathbf{R}}^*$.*

Proof. The fact that f is special Lagrangian follows immediately from the form of Ω_x given in Proposition 3.1 and the same type of calculation as performed in Corollary 1.3. Now if $y \in Y_{\tilde{\sigma},x}^{ns}$ then $\mu^{-1}(\mu(y))/T(L)$ is isomorphic to the categorical quotient $Y_{\tilde{\sigma},x}^{ns}/T_{\mathbf{C}}(L)$, which is isomorphic to \mathbf{C} with holomorphic coordinate t_0 . Thus the fibre $f^{-1}(f(y))$ is an inverse image of a straight line in \mathbf{C} under the quotient map $\mu^{-1}(\mu(y)) \rightarrow \mathbf{C}$. Thus the general fibre is $T(L) \times \mathbf{R}$. Clearly also f is surjective if μ is.

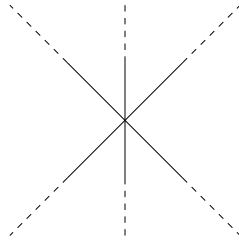
The discriminant locus Δ is the image of the union of $T(L)$ orbits of dimension $< n-1$. Now a subcone τ of dimension k of $\tilde{\sigma}$ corresponds to a codimension k orbit of $T_{\mathbf{C}}(N')$ which is fixed by $T_{\mathbf{C}}(\mathbf{R}\tau \cap N')$. Thus the $T(L)$ orbits on this stratum drop dimension if $\mathbf{R}\tau \cap L_{\mathbf{R}} \neq 0$. The one-dimensional faces of $\tilde{\sigma}$ are generated by $n \times e_k$ where n is a vertex of the polytope R_k . Thus a face τ of $\tilde{\sigma}$ has $\mathbf{R}\tau \cap L_{\mathbf{R}} \neq 0$ if and only if it contains two one-dimensional faces generated by $n_1 \times e_k$ and $n_2 \times e_k$ for some k , for n_1, n_2 two vertices of R_k . Necessarily n_1 and n_2 are the endpoints of an edge of R_k . Thus all minimal faces τ such that $\mathbf{R}\tau \cap L_{\mathbf{R}} \neq 0$ are two-dimensional faces spanned by $n_1 \times e_k$ and $n_2 \times e_k$. Now the function t_k is necessarily zero on the corresponding codimension 2 stratum. If this

codimension 2 stratum is called $D_\tau \subseteq Y_{\tilde{\sigma}}$, then on $D_\tau \cap Y_{\tilde{\sigma},x}$, $t_k - t_0 = x_k - x_0$ so $t_0 = x_0 - x_k$. Thus $f(D_\tau \cap Y_{\tilde{\sigma},x})$ is contained in the hyperplane given by $\{\text{Im}(i^{n+1}(x_0 - x_k))\} \times L_{\mathbf{R}}^*$. •

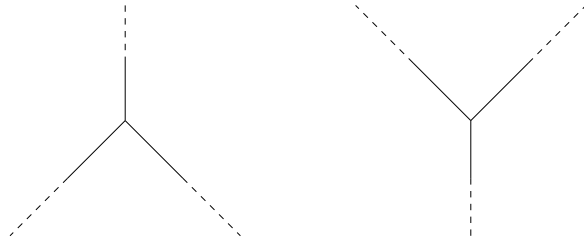
Example 3.4. Continuing with Example 3.1, the discriminant locus depends on the choice of the two decompositions. For general choice of x , the discriminant locus in the first splitting is contained in 3 different planes. There are three choices of two-dimensional τ yielding components of the discriminant locus, and for each τ , $D_\tau \cap Y_{\tilde{\sigma},x}$ (for general x , $Y_{\tilde{\sigma},x}$ is already non-singular) consists just of a \mathbf{C}^* . The image of this \mathbf{C}^* under the moment map is a straight line, and the fibres of this map are connected (by [14]). Depending on the properties of μ , this image is either a line segment, a ray, or a line infinite in both directions. So Δ looks like



where each line is in a parallel plane. As $x \rightarrow 0$, these planes will converge to the same plane, producing, for suitable choice of ω , a discriminant locus for $x = 0$ of



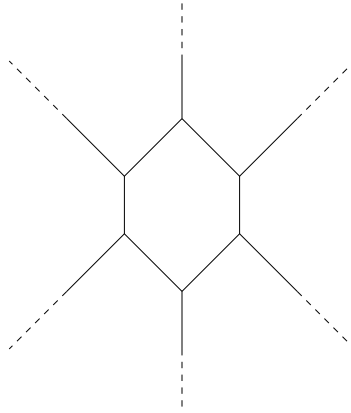
If instead we choose the second smoothing, then similar arguments show that Δ looks like



Again, we get the same picture as above as $x \rightarrow 0$.

If we take a crepant resolution of $Y_{\tilde{\sigma},0} = Y_\sigma$, then by §2 we obtain a discriminant locus

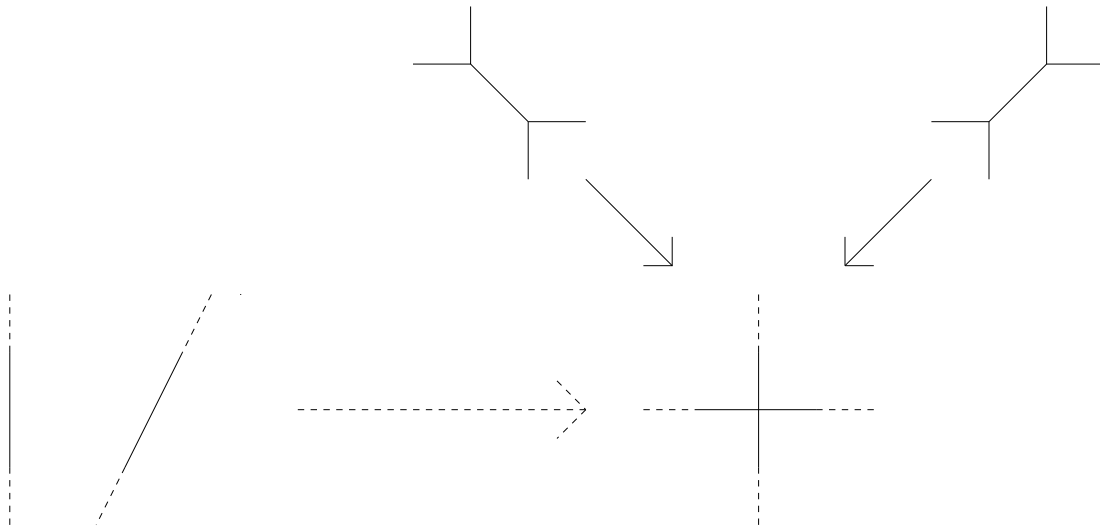
which looks like



Another, simpler, example, is the ordinary double point, given by

$$\begin{array}{c} (0,1) \\ \square \\ (0,0) \end{array} \begin{array}{c} (1,1) \\ \square \\ (1,0) \end{array} = \begin{array}{c} (0,1) \\ \text{---} \\ (0,0) \end{array} + \begin{array}{c} \text{---} \\ (0,0) \end{array} \begin{array}{c} (1,0) \end{array}$$

The discriminant locus of the smoothing consists of two lines in different planes. As $x \rightarrow 0$ these planes converge, and then there are two different small resolutions of the ordinary double point. This gives a picture



which shows how the discriminant locus changes under smoothing and resolution.

We have not discussed the choice of the metrics on these smoothings, but in some examples one can find Ricci-flat metrics. In the ordinary double point case, Stenzel [32] has constructed an explicit Ricci-flat metric, while in some examples [34,35] apply.

§4. Local mirror symmetry and connections with the work of Ruan and Joyce.

Let us begin this more informal discussion by asking the question: how do we construct the mirror to Y_Σ , where Y_Σ is as in §2? We shall focus on the three-dimensional case. This should make sense in the context of local mirror symmetry as developed in [4]. (See also related forms of local mirror symmetry in [5], [17], and [18]). If we follow the SYZ philosophy, then we would need to construct duals of the special Lagrangian fibrations $f : Y_\Sigma \rightarrow \mathbf{R}^3$ of Theorem 2.2. The difficulty is that the general fibre of f is $T^2 \times \mathbf{R}$, which we can't dualize. However, we have at least constructed, via Theorem 2.4, a topological “properification” $f' : Y'_\Sigma \rightarrow \mathbf{R}^3$ of f . (We will ignore the metric properties of Y_Σ and Y'_Σ for the moment). We can then dualize f' topologically. Indeed, from the description of the singular fibres in Remark 2.5, (2), as well as the calculation of monodromy in [8], Example 1.2, we can see that f' has only semi-stable fibres, and in fact satisfies the hypotheses of Corollary 2.2 of [10]. Thus a topological dual $\check{f} : \check{Y}_\Sigma \rightarrow \mathbf{R}^3$ of $f' : Y'_\Sigma \rightarrow \mathbf{R}^3$ exists.

However, it is worthwhile describing this dual explicitly. First, observe that the discriminant locus Δ of f' is a trivalent graph (homeomorphic to the 1-skeleton of $\partial\mu(Y_\Sigma)$). The fibres over the edges of the graph are of type (2, 2) in the notation of [9] and [10] (i.e. a product of a circle with a Kodaira type I_1 fibre) and type (1, 2) at all vertices*. Thus the dual $\check{f} : \check{Y}_\Sigma \rightarrow \mathbf{R}^3$ has only type (2, 2) and type (2, 1) fibres.

Let's describe the monodromy of the fibration f' . Fix a basis e_1, e_2, e_3 of N so that that $m_0 = e_1^* + e_2^* + e_3^*$ and take $f_1 = e_1 - e_2, f_2 = e_1 - e_3$ to be a basis for N_{m_0} . If we take any point $b \in \mathbf{R}^3 \setminus \Delta$, then the fibre $f'^{-1}(b)$ has a $T(N_{m_0})$ -action which allows us to identify N_{m_0} with a sublattice of $H_1(f'^{-1}(b), \mathbf{Z})$. We can then choose an element $f_3 \in H_1(f'^{-1}(b), \mathbf{Z})$ such that f_1, f_2, f_3 form a basis for $H_1(f'^{-1}(b), \mathbf{Z})$. It is clear f_1 and f_2 will be monodromy invariant 1-cycles.

An edge l of Δ is the image under f' of a codimension-two $T_{\mathbf{C}}(N)$ -orbit, which in turn corresponds to a dimension 2 face τ of Σ with generators n_1, n_2 . It then follows from [10], Proposition 3.3 and Example 2.8, that if b is chosen near the edge l , $\gamma : S^1 \rightarrow \mathbf{R}^3 \setminus \Delta$ a suitably oriented simple loop about l based at b , and $n_1 - n_2 = a_1 f_1 + a_2 f_2$, then the monodromy transformation $T : H_1(f'^{-1}(b), \mathbf{Z}) \rightarrow H_1(f'^{-1}(b), \mathbf{Z})$ about γ is, in the basis

* For generic good T^3 -fibrations, which only have singular fibres of type (2, 2), (2, 1) and (1, 2), I am going to second Dave Morrison's suggestion that they be called generic, negative and positive singular fibres respectively, the words negative and positive referring to the sign of the Euler characteristic of the fibre. W.-D. Ruan introduced his own notation of type II and III for these singular fibres.

$f_1, f_2, f_3,$

$$T = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The topological dual $\check{f} : \check{Y}_\Sigma \rightarrow \mathbf{R}^3$ is constructed as follows. Let $N_{m_0}^* = \text{Hom}_{\mathbf{Z}}(N_{m_0}, \mathbf{Z})$. Set $\bar{X} = T(N_{m_0}^*) \times \mathbf{R}^3$, and let $S \subseteq X$ be a topological surface constructed as follows. S projects to $\Delta \subseteq \mathbf{R}^3$, and for each edge l of Δ , $(T(N_{m_0}^*) \times l) \cap S$ is a cylinder fibering in circles over l , with the circle homotopic to $T((n_1 - n_2)^\perp) \subseteq T(N_{m_0}^*)$. At a vertex, which corresponds to a two-dimensional face of Σ , spanned by n_1, n_2 and n_3 , the classes $n_1 - n_2, n_2 - n_3$ and $n_3 - n_1$ add to zero, and thus the three cylinders can be glued above the vertex. This gives a topological manifold S .

Now let $X = \bar{X} \setminus S$, $\pi : Y \rightarrow X$ a principal S^1 -bundle with Chern class $(0, \pm 1) \in H^2(X, \mathbf{Z}) = H^2(\bar{X}, \mathbf{Z}) \oplus \mathbf{Z}$. Then there is a topological manifold \bar{Y} containing Y and a diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ \downarrow \pi & & \downarrow \bar{\pi} \\ X & \hookrightarrow & \bar{X} \end{array}$$

such that $\bar{\pi}$ is proper and the S^1 -action on Y extends to an S^1 -action on \bar{Y} , with $\bar{\pi}^{-1}(S) \cong S$ (see [10], Proposition 2.5). Taking $\check{Y}_\Sigma = \bar{Y}$ and \check{f} the composition $\bar{Y} \rightarrow \bar{X} \rightarrow \mathbf{R}^3$, we obtain the topological dual of $f' : Y'_\Sigma \rightarrow \mathbf{R}^3$. These are dual in the sense that the monodromy representations are dual, which is the only topological measure of duality.

If we are only interested in the topology, this is the end of the story. But to get further insight into the picture, let us consider the local mirror symmetry picture of [4].

One way to interpret the suggestions of [4] is as follows: the mirror to Y_Σ (not Y'_Σ !) is a curve $C \subseteq (\mathbf{C}^*)^2 = T_{\mathbf{C}}(N_{m_0}^*)$ whose Newton polygon is the translation of the polygon P into the plane $N_{m_0} \otimes \mathbf{R}$. Calculations of certain period integrals should yield the mirror map and predictions for Gromov-Witten invariants on Y_Σ . These period integrals are integrals of $\frac{dz_1 \wedge dz_2}{z_1 z_2}$ on $(\mathbf{C}^*)^2$ over 2-cycles with boundary on C . Such integrals satisfy standard Picard-Fuchs equations.

The basic claim is that the pair $S \subseteq T(N_{m_0}^*) \times \mathbf{R}^2$ contained in \bar{X} (where $\mathbf{R}^2 \subseteq \mathbf{R}^3$ is the plane containing Δ) is the same as the pair $C \subseteq T_{\mathbf{C}}(N_{m_0}^*)$. This will make the connection both between this circle of ideas and local mirror symmetry, as well as the connection with Ruan's work, clear.

To make this connection, we need to be more precise about our choice of the equation for C . First, choose a Kähler class on Y_Σ . (The choice of Kähler class determines $\Delta \subseteq \mathbf{R}^3$.)

As in §2, this can be thought of as a strictly convex function $\phi : |\Sigma| \rightarrow \mathbf{R}$, which we can restrict to the polygon P . With coordinates z_1, z_2 on $(\mathbf{C}^*)^2$ corresponding to the basis f_1, f_2 of N_{m_0} , we can consider the family of curves C_t given by the equation $h_t = 0$, $t > 0$, where

$$h_t = \sum_{(a,b) \in P \cap N_{m_0}} t^{\phi(a,b)} m_{a,b} z_1^a z_2^b,$$

with $m_{a,b} \in \mathbf{C}$. Let $\nu : T_{\mathbf{C}}(N_{m_0}^*) \rightarrow \mathbf{R}^2$ be the moment map $\nu(z_1, z_2) = (\log |z_1|, \log |z_2|)$.

The following theorem is implicit in Ruan's work ([29,30]) and can also be proved using ideas of Viro [36] and Mikhalkin [27].

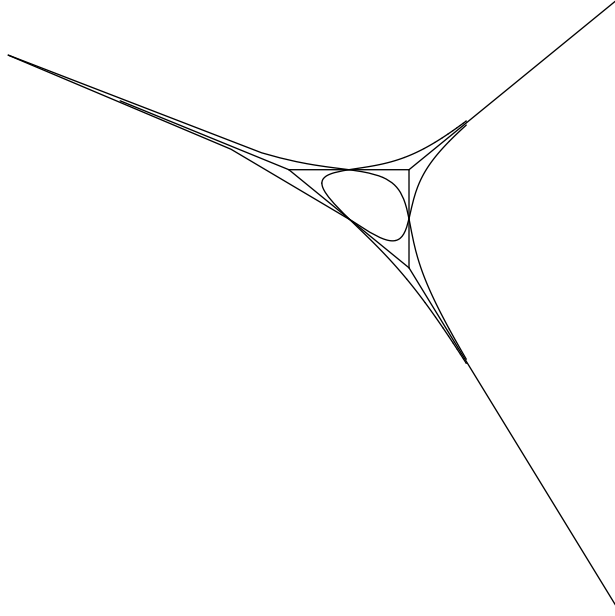
Theorem 4.1. *For $|t|$ close to zero, $\nu(C_t)$ is a fattening of the graph Δ , and there is a C^0 -isotopy of $T_{\mathbf{C}}(N_{m_0}^*) = T(N_{m_0}^*) \times \mathbf{R}^2$ identifying C_t and S .*

Ruan uses this to construct torus fibrations on toric hypersurfaces. Thus he was led to his pictures of the discriminant loci of Lagrangian fibrations by looking at the *dual* picture to the fibrations $Y_{\Sigma} \rightarrow \mathbf{R}^3$ developed in [10] and here. These two points of view complement each other nicely.

Example 4.2. Let us continue with Example 2.3. Here $m_0 = (1, 1, 1)$, and we can take a basis f_1, f_2 of N_{m_0} with $f_1 = (2/3, -1/3, -1/3)$, $f_2 = (-1/3, 2/3, -1/3)$, so that if P is translated to N_{m_0} , we can take it to be the convex hull of f_1, f_2 and $-f_1 - f_2$. The set of integral points of P then corresponds to the monomials $z_1, z_2, 1$ and $z_1^{-1} z_2^{-1}$. We can take

$$h_t = t(z_1 + z_2 + z_1^{-1} z_2^{-1}) + 1,$$

in which case, for t small, the image of C_t under ν looks like



We have in fact only drawn the boundary of $\nu(C_t)$, and superimposed the codimension two discriminant locus Δ of $f : Y_\Sigma \rightarrow \mathbf{R}^3$ with symplectic form given by the monomial-divisor mirror map. This discriminant locus Δ is lying in the interior of $\nu(C_t)$. (The actual shape of Δ disagrees with the one drawn in Example 2.3, because we have used a different basis for N_{m_0} .)

Now, so far we have produced a topological fibration $\check{f} : \check{Y}_\Sigma \rightarrow \mathbf{R}^3$. How might we construct a special Lagrangian fibration? To do this we must first realise \check{Y}_Σ as an (almost) Calabi-Yau manifold. We should expect this structure to be invariant under an S^1 action with fixed locus isomorphic to C_t . Furthermore, we might expect this S^1 action to extend to a \mathbf{C}^* action on \check{Y}_Σ . One way to accomplish this is as follows. Let h be a regular function on $(\mathbf{C}^*)^2$, and let

$$Y_h = \{(x, y, z_1, z_2) \in \mathbf{C}^2 \times (\mathbf{C}^*)^2 \mid xy = h(z_1, z_2)\}.$$

Y_h has a \mathbf{C}^* -action given by $(x, y, z_1, z_2) \mapsto (\lambda x, \lambda^{-1}y, z_1, z_2)$ for $\lambda \in \mathbf{C}^*$, and the fixed locus is the curve $x = y = h = 0$. We also need to choose a Kähler and holomorphic 3-form on Y_h . The holomorphic form will be

$$\Omega = i \frac{dx \wedge dz_1 \wedge dz_2}{xz_1z_2} = -i \frac{dy \wedge dz_1 \wedge dz_2}{yz_1z_2}$$

on Y_h . We have more choice for ω , but for convenience we will take

$$\omega = \frac{i}{2} \left(dx \wedge d\bar{x} + dy \wedge d\bar{y} + \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} + \frac{dz_2 \wedge d\bar{z}_2}{|z_2|^2} \right).$$

(The form of the part of ω which is a pull-back from $(\mathbf{C}^*)^2$ is crucial.)

Now try to construct a special Lagrangian fibration on Y_h using Theorem 1.2. The moment map of the S^1 action is $\mu : Y_h \rightarrow \mathbf{R}$ given by $\mu(x, y, z_1, z_2) = |x|^2 - |y|^2$. We need, for each $c \in \mathbf{R}$, to find a special Lagrangian fibration on $\mu^{-1}(c)/S^1$. Now $\mu^{-1}(c)/S^1$ is canonically isomorphic to $(\mathbf{C}^*)^2$ as a complex manifold, and

$$\begin{aligned} \Omega_{red} &= \iota(2i(y\partial_y - x\partial_x))\Omega \\ &= 2 \frac{dz_1 \wedge dz_2}{z_1 z_2}, \end{aligned}$$

while ω_{red} can be calculated with some effort to be

$$\omega_{red} = \frac{i}{2} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} + \frac{dz_2 \wedge d\bar{z}_2}{|z_2|^2} + \frac{1}{\sqrt{c^2 + 4|h|^2}} dh \wedge d\bar{h} \right).$$

We need to find a special Lagrangian fibration on $((\mathbf{C}^*)^2, \omega_{red}, \Omega_{red})$ depending on the value of c . Now if we took the limit $c \rightarrow \infty$, we get to the case where

$$\omega_{red} = \frac{i}{2} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} + \frac{dz_2 \wedge d\bar{z}_2}{|z_2|^2} \right),$$

and the moment map $\nu : (\mathbf{C}^*)^2 \rightarrow \mathbf{R}^2$ is then a special Lagrangian fibration. Thus the special Lagrangian fibrations for finite c , if they exist, should be viewed as a deformation of ν . For finite $c \neq 0$ one might hope to prove the existence of such a deformation using pseudo-holomorphic curve techniques and Gromov compactness. However, at $c = 0$ the almost Calabi-Yau structure becomes singular, and it is difficult to predict the structure of a special Lagrangian fibration on Y_h , if it should exist. The conjectural picture however is that there exists a special Lagrangian fibration $f : Y_h \rightarrow \mathbf{R}^3$ given by $(|x|^2 - |y|^2, g)$, where $g : Y_h \rightarrow \mathbf{R}^2$ induces a special Lagrangian fibration on each reduced space. Furthermore the discriminant locus of f will be contained in the hyperplane in \mathbf{R}^3 where the first coordinate is zero, and would be the image of $x = y = h = 0$ under the map f , i.e. the image of $h = 0$ in $(\mathbf{C}^*)^2$ under some deformation of the moment map ν .

Of course, if we are interested in a dual to $f' : Y'_\Sigma \rightarrow \mathbf{R}^3$ which is special Lagrangian, we should take $h = h_t$, identifying \check{Y}_Σ with Y_{h_t} via Theorem 4.1. Getting more speculative, we recall that dualizing should exchange information at a deeper level, i.e. interchange

the data of the symplectic structure on Y_Σ with the complex structure on \check{Y}_Σ . The precise correspondence is understood at the level of the mirror map, as calculated in [4]. I don't want to go into details here, but the main point is that one wants to compute periods of the holomorphic 3-form Ω on \check{Y}_Σ . Taking $\check{Y}_\Sigma = Y_{h_t}$, note that $H_3(\check{Y}_\Sigma, \mathbf{Z}) = H_3(\mu^{-1}(0), \mathbf{Z})$ (as gradient flow gives a retraction of \check{Y}_Σ onto $\mu^{-1}(0)$), and in turn $H_3(\mu^{-1}(0), \mathbf{Z}) \cong H_2((\mathbf{C}^*)^2, C_t, \mathbf{Z})$. In other words a 3-cycle is the inverse image under the quotient map of a 2-chain with boundary in C_t . Integrating over such a 3-cycle reduces to integrating $\frac{dz_1 \wedge dz_2}{z_1 z_2}$ over such a 2-chain. These period integrals satisfy the relevant Picard-Fuchs equations and define the right mirror map as explored in [4]. This perhaps explains why $C_t \subseteq (\mathbf{C}^*)^2$ should be considered as the mirror of Y_Σ .

As one further intuitive observation in this direction, let us try to explain why it makes sense to consider $C_t \subseteq (\mathbf{C}^*)^2$ to be the mirror of Y_Σ rather than Y'_Σ . T -duality, to first approximation, should exchange long and short distances in the fibres. Thus the fibres of $f : Y_\Sigma \rightarrow \mathbf{R}^3$ should be viewed as a limit of tori with greater and greater radius in one direction, so in the limit the fibre is $\mathbf{R} \times T^2$ rather than T^3 . The T -dual fibres should have this radius approaching zero. Intuitively, it then appears natural to divide Y_{h_t} by the S^1 -action, as the S^1 's should correspond to the “small” direction.

This discussion should not be taken too seriously. It is clear that there is much to understand, but I believe this circle of ideas and examples will prove to be an excellent laboratory for exploring the more intricate questions surrounding the SYZ conjecture.

Perhaps the most pressing question is the following. It was originally my and many others' hope that special Lagrangian fibrations would be reasonably differentiable, and differentiability implies certain conditions on the discriminant locus. For example, the discriminant locus is Hausdorff codimension 2 if the map is C^∞ (see [10], §1). However, Joyce has now given in [22] examples of special Lagrangian fibrations which are only piecewise differentiable, and whose discriminant locus is codimension one. In fact, his basic example is of a very similar flavour to the S^1 -invariant setup above. From these examples, I believe the likelihood is that if $f : Y_h \rightarrow \mathbf{R}^3$ exists, it is only piecewise smooth, and the discriminant locus is amoeba-like rather than a graph. Thus if there is a special Lagrangian fibration $f : Y_{h_t} \rightarrow \mathbf{R}^3$, it is only a perturbation of the topological fibration $\check{f} : \check{Y}_\Sigma \rightarrow \mathbf{R}^3$.

§5. The Future of the SYZ Conjecture.

It is clear that Joyce's picture forces us to reconsider the full strength version of the SYZ conjecture, as opposed to the topological ones considered in [10] and [29]. The

notion of dualizing topological torus fibrations as developed in [10] will not be the right one. We would expect that if a mirror pair X, \check{X} possess special Lagrangian fibrations $f : X \rightarrow B, \check{f} : \check{X} \rightarrow B$, they will have different amoeba-like discriminant loci Δ and $\check{\Delta}$. They will presumably be dual only in a relatively crude topological sense in that the monodromy representations $\rho : \pi_1(B \setminus \Delta) \rightarrow SL_3(\mathbf{Z})$ and $\check{\rho} : \pi_1(B \setminus \check{\Delta}) \rightarrow SL_3(\mathbf{Z})$ are dual representations.

It is my current belief that the SYZ conjecture will make most sense in a limiting picture. First let us recall from [16] certain structures which appear naturally on moduli spaces of special Lagrangian submanifolds. Let (X, ω, Ω) be a Calabi-Yau manifold, B a moduli space of deformations of some special Lagrangian submanifold on X , along with a universal family

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & X \times B \\ \downarrow f & & \\ B & & \end{array}$$

Let $p : \mathcal{U} \rightarrow X$ be the projection. All fibres of f are assumed to be smooth submanifolds of X . Then from [26], we know B is smooth, with a canonical identification of $\mathcal{T}_{B,b}$ with $\mathcal{H}^1(f^{-1}(b), \mathbf{R})$, the space of \mathbf{R} -valued harmonic one-forms, on $f^{-1}(b)$. This is via the map $v \in \mathcal{T}_{B,b} \mapsto \iota(v)p^*\omega$, where v is pulled back to a vector field normal to $f^{-1}(b)$ in \mathcal{U} .

There are two important structures on B :

- (1) an integral affine structure. If $U \subseteq B$ is a contractible open set, with coordinates t_1, \dots, t_n , let $\gamma_1, \dots, \gamma_n \in H_1(f^{-1}(b), \mathbf{Z})$ be a basis for first homology varying continuously with b . Then the 1-forms α_i given by

$$\partial/\partial t_j \mapsto \int_{\gamma_i} \iota(\partial/\partial t_j)p^*\omega$$

on B are closed and linearly independent ([16], Proposition 1). Thus there exists a coordinate system y_1, \dots, y_n on U with $\alpha_i = dy_i$, and these coordinates are well-defined up to integral affine transformations (elements of $\mathbf{R}^n \rtimes GL_n(\mathbf{Z})$). This defines an integral affine structure on B .

- (2) There is a metric on B (the McLean metric) given by

$$g(\partial/\partial t_i, \partial/\partial t_j) = - \int_{f^{-1}(b)} \iota(\partial/\partial t_i)p^*\omega \wedge \iota(\partial/\partial t_j)p^*\text{Im } \Omega.$$

(One might want to normalize this metric in various ways).

Hitchin showed there is a compatibility between the metric and affine structure: locally there exists a function K such that $g(\partial/\partial y_i, \partial/\partial y_j) = \partial^2 K / \partial y_i \partial y_j$. Kontsevich and

Soibelman [23] call this structure on B_0 of affine structure plus metric of this form an *affine Kähler* (AK) manifold. Such a manifold was called *Hessian* in earlier work of H. Shima: see [31] and references therein. If in addition the function K satisfies the real Monge-Ampère equation $\det \partial^2 K / \partial y_i \partial y_j = \text{constant}$, Kontsevich and Soibelman call such an Hessian manifold a *Monge-Ampère manifold*.

Next recall the definition of Gromov-Hausdorff convergence.

Definition 5.1. *Let $(X, d_X), (Y, d_Y)$ be two compact metric spaces. Suppose there exists maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ (not necessarily continuous) such that for all $x_1, x_2 \in X$,*

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \epsilon$$

and for all $x \in X$,

$$d_X(x, g \circ f(x)) < \epsilon,$$

and the two symmetric properties for Y hold. Then we say the Gromov-Hausdorff distance between X and Y is at most ϵ . The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of all such ϵ .

There are two distinct situations we might want to apply this notion. The first was discussed independently by myself and Wilson [11] and Kontsevich and Soibelman in [23]. The second situation follows naturally from these ideas. First, let $\mathcal{X} \rightarrow \Delta$ be a flat family of degenerating Calabi-Yau n -folds, with $0 \in \Delta$ a large complex structure limit point (or maximally unipotent boundary point). Let $t_i \in \Delta$ be a sequence of points converging to $0 \in \Delta$, and let g_i on \mathcal{X}_{t_i} be a Ricci-flat metric normalized so that $\text{Diam}(\mathcal{X}_{t_i}, g_i)$ remains constant. Then general results about Gromov-Hausdorff convergence tell us that a subsequence of (\mathcal{X}_{t_i}, g_i) converges to a metric space (X_∞, g_∞) .

Second, consider another sequence of metric spaces, whose existence (or rather non-emptiness) is currently conjectural. Suppose that for t_i sufficiently close to 0, there is a special Lagrangian T^n whose homology class is invariant under monodromy near 0. (This is a property we expect to find of fibres of a special Lagrangian fibration associated to a large complex structure limit point). Let $B_{0,i}$ be the moduli space of deformations of this torus, every point of $B_{0,i}$ corresponding to a smooth torus in \mathcal{X}_{t_i} . The manifold $B_{0,i}$ comes equipped with the McLean metric. We should then compactify $B_{0,i} \subseteq B_i$ in some manner: probably taking the closure of $B_{0,i}$ in the space of special Lagrangian currents on \mathcal{X}_{t_i} is the right thing to do. This should give a series of metric spaces (B_i, d_i) , which again, if the McLean metric is normalized properly to keep the diameter constant, may have a convergent subsequence, converging to a compact metric space (B_∞, d_∞) .

The following is a slight souping up of the conjectures in [11] and [23].

Conjecture 5.2. *If (\mathcal{X}_{t_i}, g_i) converges to (X_∞, g_∞) and (B_i, d_i) is non-empty for large i and converges to (B_∞, d_∞) , then B_∞ and X_∞ are isometric up to scaling. Furthermore, there is a subspace $B_0 \subseteq B_\infty$ with $\Delta = B_\infty \setminus B_0$ of Hausdorff codimension 2 in B_∞ such that B_0 is a Monge-Ampère manifold, with the metric inducing d_∞ on B_0 .*

This is a considerably weaker conjecture than the original full-strength SYZ proposal on the existence of special Lagrangian fibrations. But following the philosophy of Kontsevich and Soibelman, this should be sufficient for most purposes.

Remarks 5.2. (1) This conjecture doesn't assume the existence of special Lagrangian fibrations on \mathcal{X}_{t_i} for any i . It would of course be nice if this is the case, but taking Joyce's philosophy seriously means we only see the codimension 2 structure in the limit. We expect that as $i \rightarrow \infty$, the area of the critical locus of a special Lagrangian fibration on \mathcal{X}_{t_i} goes to zero, so its image hopefully deforms to something of codimension two.

Even once one finds a single special Lagrangian torus, it could fail to give a fibration either because deformations may not be disjoint from each other, or the deformations simply may not fill out the entire manifold, so that B_i has a boundary. The expectation might be that these sorts of things are more likely to happen near the discriminant locus in B_∞ .

(2) We do not expect $B_{0,i}$ to be a Monge-Ampère manifold, but only an affine Kähler manifold. This is because of the existence of examples of moduli of special Lagrangian tori where this is not the case: see the work of Matessi in [25].

(3) Stated in the proper way, [11] proves this conjecture for K3 surfaces.

The philosophy of Kontsevich and Soibelman, which I believe is the right one, is that it may be enough to work purely with the limiting data. Whereas the original form of the SYZ conjecture proposed dualizing torus fibrations, we instead dualize the limiting data. Given a Hessian manifold, one obtains a new affine structure with local affine coordinates $\tilde{y}_i = \partial K / \partial y_i$, where K is the potential of the metric. The metric remains the same, but the new potential \tilde{K} is the Legendre transform of K . This was first suggested in the context of mirror symmetry by Hitchin in [16], and this idea was used effectively in [23] and [24][†].

We are left with two fundamental questions:

The Limit Question. *How can one calculate the limit data, or guess it conjecturally*

[†] Intriguingly, this duality was mentioned in [31], which gave a reference to a 1985 work in statistics, [2], which makes serious use of this duality between Hessian manifolds.

either for general degenerations or for standard cases such as hypersurfaces in toric varieties?

The Reconstruction Question. *Given a set of limiting data, how do we reconstruct a family of Calabi-Yau manifolds converging to this limit?*

Kontsevich and Soibelman discuss these two questions, suggesting some approaches involving rigid analytic geometry and Berkovich spaces. From my point of view, these questions can be developed at a topological level (where one only preserves the limiting information of monodromy about Δ), a symplectic level (pay attention only to the affine structure) and the full metric level. For the limit question, [10] gives a conjectural limit for the quintic and its mirror on the topological level, while [29] gives it for general toric hypersurfaces. The affine structure can be guessed at from the ideas in [29], and in future work I will give a purely combinatorial description of a conjectural affine structure in the limit for hypersurfaces in toric varieties.

For the reconstruction question, the results of [10] allow a reconstruction of the underlying topological manifold from the limiting data in sufficiently generic cases, while current work in progress of my own explores the symplectic reconstruction problem.

However, solving these general questions at the metric (or complex structure) level will require some substantial new ideas.

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